

Multi-particle Schrödinger operators with point interactions in the plane

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2004 J. Phys. A: Math. Gen. 37 9157

(<http://iopscience.iop.org/0305-4470/37/39/008>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.64

The article was downloaded on 02/06/2010 at 19:12

Please note that [terms and conditions apply](#).

Multi-particle Schrödinger operators with point interactions in the plane

J Dimock¹ and S G Rajeev²

¹ Department of Mathematics, SUNY at Buffalo, Buffalo, NY 14260, USA

² Department of Physics and Astronomy, University of Rochester, Rochester, NY 14627, USA

Received 29 April 2004, in final form 29 June 2004

Published 15 September 2004

Online at stacks.iop.org/JPhysA/37/9157

doi:10.1088/0305-4470/37/39/008

Abstract

We study a system of N -bosons in the plane interacting with delta function potentials. After a coupling constant renormalization we show that the Hamiltonian defines a self-adjoint operator and obtain a lower bound for the energy. The same results hold if one includes a regular inter-particle potential.

PACS numbers: 03.65.Db, 03.65.Ge

1. Introduction

We consider N -bosons of mass m in \mathbb{R}^2 interacting with delta function potentials of strength g . The Hamiltonian for the system is

$$H = \sum_{i=1}^N \frac{-\Delta}{2m} - \sum_{1 \leq i < j \leq N} g \delta(x_i - x_j) \quad (1)$$

defined on $\mathcal{H}_N =$ the N -fold symmetric tensor product of $\mathcal{L}^2(\mathbb{R}^2)$ with itself. The problem is to make sense of this as a self-adjoint operator. This is necessary in order that the global dynamics $\psi_t = e^{-iHt} \psi_0$ be well defined. However, the expression is quite singular and one finds that the coupling constant g must be renormalized to have a chance of success.

The problem is fairly well understood for $N = 2$. We give a treatment below which involves introducing a momentum cutoff, choosing a cutoff-dependent coupling constant and then showing that as the cutoff is removed the Hamiltonians have a self-adjoint limit in the sense of resolvent convergence.

The $N = 2$ problem has also been considered by Albeverio *et al* [1]. They take a different approach which involves specifying boundary conditions when the points coincide. We show that our results are equivalent to theirs.

Our main interest is in general N and the challenge is to incorporate the wisdom gained for the two-particle case into the multi-particle setting. Our solution involves introducing fictitious particles known as *angels* which serve as markers for two-particle subsystems. This approach

was developed by one of us in the papers [2, 3]. The present paper is a rigorous version of this work. The main result is a proof that for any N the cutoff-renormalized operators have a self-adjoint limit in the sense of resolvent convergence. We also obtain a lower bound for the Hamiltonian.

This problem was previously considered by Dell'Antonio *et al* [4]³. Our results are very similar to theirs; however the proofs are rather different. They use a concept of Γ -convergence rather than resolvent convergence. Also their 'bare coupling constant' depends on N and momentum as well as the cutoff, whereas ours depends only on the cutoff.

The present work seems to have some advantages in simplicity and flexibility compared to [4]. As evidence of this we obtain the new result that essentially the same conclusions hold if we include a regular inter-particle potential in addition to the delta function.

2. Two particles

2.1. Point interaction

We begin with a discussion of the case $N = 2$. Taking mass $m = 1$ and passing to centre of mass coordinates we have the Hamiltonian

$$H = -\Delta - g\delta \quad (2)$$

on the space $\mathcal{L}^2(\mathbb{R}^2)$. In momentum space

$$(H\psi)(p) = p^2\psi(p) - \frac{g}{(2\pi)^2} \int \psi(q) \, dq. \quad (3)$$

This operator does not map into $\mathcal{L}^2(\mathbb{R}^2)$ and cannot determine a dynamics as such.

Instead we consider approximate Hamiltonians

$$(H_\Lambda\psi)(p) = p^2\psi(p) - \frac{g_\Lambda}{(2\pi)^2} \rho_\Lambda(p) \int \rho_\Lambda(q)\psi(q) \, dq \quad (4)$$

where ρ_Λ is the characteristic function of $|p| \leq \Lambda$. We define

$$P_f\psi = f(f, \psi). \quad (5)$$

(If $\|f\| = 1$ this is the projection onto f .) Then we can write with $H_0 = p^2$

$$H_\Lambda = H_0 - \frac{g_\Lambda}{(2\pi)^2} P_{\rho_\Lambda}. \quad (6)$$

This is a bounded perturbation of the self-adjoint operator H_0 and so is self-adjoint on $D(H_0)$ (Kato's theorem [6]). We define the resolvents

$$R_0(E) = (H_0 - E)^{-1} \quad R_\Lambda(E) = (H_\Lambda - E)^{-1} \quad (7)$$

when they exist. If they exist as bounded operators one says that E is in the resolvent set of the operator. The resolvent set for H_0 is $\mathbb{C} - [0, \infty)$. Since the perturbation is rank one the resolvent $R_\Lambda(E)$ can be explicitly calculated. For $E \in \mathbb{C} - [0, \infty)$ one finds that E is in the resolvent set for $H_\Lambda - E$ if and only if

$$(2\pi)^2 g_\Lambda^{-1} \neq (\rho_\Lambda, R_0(E)\rho_\Lambda) \quad (8)$$

in which case

$$R_\Lambda(E) = R_0(E) + \left(\frac{1}{(2\pi)^2 g_\Lambda^{-1} - (\rho_\Lambda, R_0(E)\rho_\Lambda)} \right) P_{R_0(E)\rho_\Lambda}. \quad (9)$$

³ They also consider the three-dimensional case. See also the early work of Flammand [5].

Indeed if (8) holds then an explicit calculation shows that the right-hand side provides a bounded inverse for $H_\Lambda - E$. On the other hand, if $(2\pi)^2 g_\Lambda^{-1} = (\rho_\Lambda, R_0(E)\rho_\Lambda)$ then

$$(H_\Lambda - E)R_0(E)\rho_\Lambda = \left(1 - \frac{g_\Lambda}{(2\pi)^2}(\rho_\Lambda, R_0(E)\rho_\Lambda)\right)\rho_\Lambda = 0 \quad (10)$$

and $R_0(E)\rho_\Lambda \neq 0$ so E is an eigenvalue of $H_\Lambda - E$ and not in the resolvent set.

Now we introduce a new parameter $\mu > 0$ and make the choice

$$g_\Lambda = g_\Lambda(\mu) = (2\pi)^2 \left(\int_{|p| \leq \Lambda} (p^2 + \mu^2)^{-1} dp \right)^{-1}. \quad (11)$$

Thus g_Λ goes to zero logarithmically as $\Lambda \rightarrow \infty$. Then we can write

$$R_\Lambda(E) = R_0(E) + \xi_\Lambda(\mu^2, -E)^{-1} P_{R_0(E)\rho_\Lambda} \quad (12)$$

where

$$\xi_\Lambda(a, b) \equiv \int_{|p| \leq \Lambda} (p^2 + a)^{-1} dp - \int_{|p| \leq \Lambda} (p^2 + b)^{-1} dp. \quad (13)$$

For $a, b > 0$ we have

$$\begin{aligned} \xi_\Lambda(a, b) &\equiv \pi \log\left(\frac{\Lambda^2}{a} + 1\right) - \pi \log\left(\frac{\Lambda^2}{b} + 1\right) \\ &= \pi \log\left(\frac{1}{a} + \frac{1}{\Lambda^2}\right) - \pi \log\left(\frac{1}{b} + \frac{1}{\Lambda^2}\right). \end{aligned} \quad (14)$$

In the last step we have cancelled the divergence in each term by adding and subtracting $\pi \log \Lambda^2$. Now it is a simple matter to take the limit $\Lambda \rightarrow \infty$ and get

$$\xi(a, b) = \pi \log(b/a). \quad (15)$$

Theorem 1.

1. For E real and not in $\{-\mu^2\} \cup [0, \infty)$ the strong limit $R(E) = \lim_{\Lambda \rightarrow \infty} R_\Lambda(E)$ exists and is given by

$$R(E) = R_0(E) + \xi(\mu^2, -E)^{-1} P_{\Omega_E} \quad (16)$$

where $\Omega_E \in \mathcal{L}^2(\mathbb{R}^2)$ is defined by

$$\Omega_E(p) = (p^2 - E)^{-1}. \quad (17)$$

2. $R(E)$ is invertible
3. For E complex and not in $\{-\mu^2\} \cup [0, \infty)$ the limit $R(E) = \lim_{\Lambda \rightarrow \infty} R_\Lambda(E)$ exists. There is a self-adjoint operator $H(\mu)$ such that $R(E) = (H(\mu) - E)^{-1}$.

Proof.

1. Under our hypotheses $\xi(\mu^2, -E) = \pi \log(-E/\mu^2) \neq 0$. Hence $\xi_\Lambda(-E, \mu^2) \neq 0$ for Λ sufficiently large and $\xi(\mu^2, -E)^{-1} = \lim_{\Lambda \rightarrow \infty} \xi_\Lambda(\mu^2, -E)^{-1}$. We also have in $\mathcal{L}^2(\mathbb{R}^2)$ the limit $\Omega_E = \lim_{\Lambda \rightarrow \infty} R_0(E)\rho_\Lambda$. The result follows.
2. To show the null space of $R(E)$ is $\{0\}$ it is sufficient to find a dense set $\mathcal{D} \subset D(H_\Lambda)$ such that for $\eta \in \mathcal{D}$ we have the existence of $\eta^* = \lim_{\Lambda \rightarrow \infty} (H_\Lambda - E)\eta$. For then if $R(E)\psi = 0$ we have

$$(\eta, \psi) = \lim_{\Lambda \rightarrow \infty} ((H_\Lambda - E)\eta, R_\Lambda(E)\psi) = (\eta^*, R(E)\psi) = 0 \quad (18)$$

for all $\eta \in \mathcal{D}$ and hence $\psi = 0$.

For our domain \mathcal{D} we pick $u \in \mathcal{S}(\mathbb{R}^2)$ so the the Fourier transform \hat{u} is in $\mathcal{C}_0^\infty(\mathbb{R}^2 - \{0\})$. For u in this domain we have

$$(H_\Lambda u)(p) = p^2 u(p) - \frac{g_\Lambda}{(2\pi)^2} \rho_\Lambda(p) \int \rho_\Lambda(q) u(q) \, dq. \tag{19}$$

We argue that the second term converges to zero so that $H_\Lambda u \rightarrow H_0 u$. Since $g_\Lambda \rightarrow 0$ and $\|\rho_\Lambda\| = \sqrt{\pi} \Lambda$ it suffices that $\int \rho_\Lambda(q) u(q) \, dq = \mathcal{O}(\Lambda^{-1})$.

To see this first replace $\rho_\Lambda(q) = \rho_1(q/\Lambda)$ by $\rho_\Lambda^*(q) = \rho^*(q/\Lambda)$ where ρ^* is smooth approximation to the characteristic function of the unit disc. The difference is $\mathcal{O}(\Lambda^{-n})$ for any n , and so it suffices to show $\int \rho_\Lambda^*(q) u(q) \, dq = \mathcal{O}(\Lambda^{-1})$.

Since $\hat{u} \in \mathcal{C}_0^\infty(\mathbb{R}^2 - \{0\})$ we have $\hat{v}(x) = |x|^{-2} \hat{u}(x)$ in the same space and so $u = -\Delta_q v$ for some $v \in \mathcal{S}(\mathbb{R}^2)$. Then after integrating by parts

$$\int \rho_\Lambda^*(q) u(q) \, dq = \int (-\Delta_q \rho_\Lambda^*)(q) v(q) \, dq. \tag{20}$$

This is $\mathcal{O}(\Lambda^{-2})$ since $|\Delta_q \rho_\Lambda^*(q)|$ is $\mathcal{O}(\Lambda^{-2})$ and $v(q)$ is rapidly decreasing.

3. This follows from the first two parts and a version of the Trotter–Kato theorem quoted in the appendix. □

Remarks.

1. The resolvent has a simple pole at $E = -\mu^2$ so $H(\mu)$ has the eigenvalue $-\mu^2$. The residue is the projection onto the eigenspace which we see is spanned by $\Omega_{-\mu^2}(p) = (p^2 + \mu^2)^{-1}$. This is a bound state.
2. Our approach to this problem follows a path well known to physicists. The problem is usually cited as an example of *dimensional transmutation* in which a model without a length scale (the coupling constant g is dimensionless) upon renormalization gains a length scale (namely μ^{-1}) [7]. This phenomenon is expected to occur in gauge theories in four dimensions.
3. Let us compare our result with the result of Albeverio *et al* [1]. They consider $-\Delta$ on $\mathcal{L}^2(\mathbb{R}^2 \setminus \{0\})$ and obtain all possible self-adjoint extensions by imposing boundary conditions at the origin. The self-adjoint extensions are indexed by a parameter α taking all real values. They also have an explicit formula for the resolvent (a Krein formula) which is just like our equation (16) except that instead of $\xi(\mu^2, -E) = \pi \log(-E/\mu^2)$ they have the function (p 99, equation (5.16))

$$4\pi^2 \alpha - 2\pi \Psi(1) + \pi \log(-E/4). \tag{21}$$

Comparing, we see that they agree exactly if the parameters are related by

$$\log \mu = -2\pi \alpha + \Psi(1) + \log 2. \tag{22}$$

2.2. Extension

The previous results can be generalized to allow an additional potential besides the delta function. We consider

$$H^\# = -\Delta + v - g\delta. \tag{23}$$

For simplicity we will assume v is a bounded function on \mathbb{R}^2 . To define this we again start with approximate Hamiltonians in momentum space

$$H_\Lambda^\# = H_0 + v' - (2\pi)^{-2} g_\Lambda P_{\rho_\Lambda} \tag{24}$$

where $g_\Lambda = g_\Lambda(\mu)$ is as before and $v' = \mathcal{F}v\mathcal{F}^{-1}$ is a convolution operator (\mathcal{F} = Fourier transform). Since $\|v'\| = \|v\| = \|v\|_\infty$ this is still a bounded perturbation and so $H_\Lambda^\#$ is self-adjoint on $D(H_0)$. Without the approximate delta function we have

$$H_1 = H_0 + v' \quad (25)$$

which is also self-adjoint on $D(H_0)$ and satisfies and $H_1 \geq -\|v\|_\infty$.

Resolvents are denoted by

$$R_1(E) = (H_1 - E)^{-1} \quad R_\Lambda^\#(E) = (H_\Lambda^\# - E)^{-1}. \quad (26)$$

If E is complex and not in $[-\|v\|_\infty, \infty)$ then E is in the resolvent set for H_1 . As before we find that such E are also in the resolvent set for $H_\Lambda^\#$ if and only if $(2\pi)^2 g_\Lambda^{-1} \neq (\rho_\Lambda, R_1(E)\rho_\Lambda)$ in which case

$$R_\Lambda^\#(E) = R_1(E) + \left(\frac{1}{(2\pi)^2 g_\Lambda^{-1} - (\rho_\Lambda, R_1(E)\rho_\Lambda)} \right) P_{R_1(E)\rho_\Lambda}. \quad (27)$$

Theorem 2.

1. For real $E < -e_0$ with

$$e_0 = \max(\|v\|_\infty + 1, \mu^2 e^{\|v\|_\infty + 1}) \quad (28)$$

the strong limit $R^\#(E) = \lim_{\Lambda \rightarrow \infty} R_\Lambda^\#(E)$ exists.

2. $R^\#(E)$ is invertible.

3. $R^\#(E) = \lim_{\Lambda \rightarrow \infty} R_\Lambda^\#(E)$ exists for all complex E not in $[-e_0, \infty)$. There is a self-adjoint operator $H^\#(\mu)$ satisfying $H^\#(\mu) \geq -e_0$ such that

$$R^\#(E) = (H^\#(\mu) - E)^{-1}. \quad (29)$$

Proof. In the denominator in (27) we insert

$$R_1(E) = R_0(E) - R_1(E)v'R_0(E) \quad (30)$$

and find

$$R_\Lambda^\#(E) = R_1(E) + \left(\frac{1}{\xi_\Lambda(\mu^2, -E) + (\rho_\Lambda, R_1(E)v'R_0(E)\rho_\Lambda)} \right) P_{R_1(E)\rho_\Lambda}. \quad (31)$$

As $\Lambda \rightarrow \infty$ we have in $\mathcal{L}^2(\mathbb{R}^2)$

$$\begin{aligned} \lim_{\Lambda \rightarrow \infty} R_1(E)\rho_\Lambda &= \lim_{\Lambda \rightarrow \infty} R_0(E)\rho_\Lambda - R_1(E)v'R_0(E)\rho_\Lambda \\ &= \Omega_E - R_1(E)v'\Omega_E \\ &\equiv \Omega_{1,E}. \end{aligned} \quad (32)$$

Thus we have the limit $R^\#(E) = \lim_{\Lambda \rightarrow \infty} R_\Lambda^\#(E)$ given by

$$R^\#(E) = R_1(E) + \left(\frac{1}{\xi(\mu^2, -E) + (\Omega_E, v'\Omega_E) - (\Omega_E, v'R_1(E)v'\Omega_E)} \right) P_{\Omega_{1,E}} \quad (33)$$

provided the denominator does not vanish. However $\|\Omega_E\|_2^2 = \pi|E|^{-1}$ and since $E < -\|v\|_\infty - 1$ we have $\|R_1(E)\| \leq 1$ and hence

$$\begin{aligned} |(\Omega_E, v'\Omega_E)| &\leq \pi|E|^{-1}\|v\|_\infty \leq \pi \\ |(\Omega_E, v'R_1(E)v'\Omega_E)| &\leq \pi|E|^{-1}\|v\|_\infty^2 \leq \pi\|v\|_\infty. \end{aligned} \quad (34)$$

Thus we can avoid vanishing provided $\xi(\mu^2, -E) > \pi(\|v\|_\infty + 1)$ or $\log(-E/\mu^2) > \|v\|_\infty + 1$. This is our condition $-E > \mu^2 e^{\|v\|_\infty + 1}$.

Thus part one is proved. The second and third parts follow as in the previous theorem. \square

3. Many particles

3.1. Bosons

We now turn to the many-particle problem. It is convenient to work with all possible values of N at the same time, even though the main interest is at fixed N . This means we are working on the Fock space $\mathcal{H} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N$. This has the usual creation and annihilation operators $a^*(f), a(f)$ defined for $f \in \mathcal{L}^2(\mathbb{R}^2)$. We also have $a(p) = a(\delta(\cdot - p))$ defined on the domain \mathcal{D} which is the dense subspace of \mathcal{H} with only a finite number of entries and wavefunctions in the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. For $\psi \in \mathcal{D}$ the function $p \rightarrow a(p)\psi$ is rapidly decreasing. (Note that $a^*(p) = a^*(\delta(\cdot - p))$ is not an operator.)

The Hamiltonian has the form $H = H_0 + H_I$. The free Hamiltonian H_0 is $\sum_{i=1}^N p_i^2/2$ on the \mathcal{H}_N and is essentially self-adjoint on $\mathcal{D} \cap \mathcal{H}_N$. It can also be represented as a bilinear form on $\mathcal{D} \times \mathcal{D}$ as⁴

$$H_0 = \int \omega(p)a^*(p)a(p) dp \quad \omega(p) = \frac{p^2}{2}. \tag{35}$$

The interaction with inter-particle potential $-g\delta(x - y)$ is given in momentum space by the bilinear form on $\mathcal{D} \times \mathcal{D}$:

$$H_I = \frac{-g}{2(2\pi)^2} \int a^*(p'_1)a^*(p'_2)\delta(p_1 + p_2 - p'_1 - p'_2)a(p_1)a(p_2) dp_1 dp_2 dp'_1 dp'_2. \tag{36}$$

However, this is not an operator.

To remedy this we introduce

$$H_{I,\Lambda} = H_0 + H_{I,\Lambda}. \tag{37}$$

For $H_{I,\Lambda}$ we add momentum cutoffs ρ_Λ , take the coupling constant $g_\Lambda = g_\Lambda(\mu)$ as before and define

$$H_{I,\Lambda} = \frac{-g_\Lambda}{2(2\pi)^2} \int \rho_\Lambda\left(\frac{p_1 - p_2}{2}\right) \rho_\Lambda\left(\frac{p'_1 - p'_2}{2}\right) \times a^*(p'_1)a^*(p'_2)\delta(p_1 + p_2 - p'_1 - p'_2)a(p_1)a(p_2) dp_1 dp_2 dp'_1 dp'_2. \tag{38}$$

Changing variables to

$$p = p_1 + p_2 \quad q = \frac{p_1 - p_2}{2} \tag{39}$$

we find for the associated quadratic form

$$(\psi, H_{I,\Lambda}\psi) = \frac{-g_\Lambda}{2(2\pi)^2} \int \rho_\Lambda(q)\rho_\Lambda(q') \left(a\left(\frac{p}{2} + q'\right) a\left(\frac{p}{2} - q'\right) \times \psi, a\left(\frac{p}{2} + q\right) a\left(\frac{p}{2} - q\right) \psi \right) dp dq dq'. \tag{40}$$

Applying the Schwarz inequality first in Fock space and then in the integral we find

$$\begin{aligned} |(\psi, H_{I,\Lambda}\psi)| &\leq \frac{g_\Lambda}{2(2\pi)^2} \left(\int \rho_\Lambda(q)^2 dq \right) \int \left\| a\left(\frac{p}{2} + q\right) a\left(\frac{p}{2} - q\right) \psi \right\|^2 dp dq \\ &= \frac{g_\Lambda}{2(2\pi)^2} \|\rho_\Lambda\|_2^2 \int \|a(p_1)a(p_2)\psi\|^2 dp_1 dp_2 \\ &= \frac{g_\Lambda}{2(2\pi)^2} \|\rho_\Lambda\|_2^2 \|N_0^{1/2}(N_0 - 1)^{1/2}\psi\|^2. \end{aligned} \tag{41}$$

Here $N_0 = \int a^*(p)a(p) dp$ is the number operator.

⁴ This means $(\phi, H_0\psi) = \int \omega(p)(a(p)\phi, a(p)\psi) dp$ or as a quadratic form $(\psi, H_0\psi) = \int \omega(p)\|a(p)\psi\|^2 dp$.

On the N -particle subspace \mathcal{H}_N we have $N_0 = N$ and hence $H_{I,\Lambda}$ is a bounded quadratic form. This determines a bounded self-adjoint operator on each \mathcal{H}_N and hence H_Λ defines a self-adjoint operator on each \mathcal{H}_N with domain $D(H_0) \cap \mathcal{H}_N$. Taking the direct sum we get a self-adjoint operator H_Λ on the full Fock space.

3.2. Angels

Next we introduce angels. We define

$$\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}^2) \otimes \mathcal{H} \quad (42)$$

which is Fock space with an angel. For $f \in \mathcal{L}^2(\mathbb{R}^2)$ we define $\chi(f) : \tilde{\mathcal{H}} \rightarrow \mathcal{H}$ and $\chi^*(f) : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ by

$$\chi(f)(h \otimes \psi) = (f, h)\psi \quad \chi^*(f)\psi = f \otimes \psi. \quad (43)$$

These are creation and annihilation operators for angels, are adjoint to each other and satisfy

$$\chi(f)\chi^*(h) = (f, h) \quad \chi^*(h)\chi(f) = h(f, \cdot) \otimes I. \quad (44)$$

There is also the operator $\chi(p) = \chi(\delta(\cdot - p))$ defined say on the dense subspace $\tilde{\mathcal{D}} \subset \tilde{\mathcal{H}}$ defined by $\tilde{\mathcal{D}} \equiv \mathcal{S}(\mathbb{R}^2) \otimes \mathcal{D}$.

An equivalent representation is

$$\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}^2, \mathcal{H}). \quad (45)$$

Then $\tilde{\mathcal{D}}$ is a subspace of $\mathcal{S}(\mathbb{R}^2, \mathcal{D})$ and on this domain

$$\chi(p)\Psi = \Psi(p). \quad (46)$$

Next we introduce

Definition 1.

$$B_\Lambda = \frac{1}{\sqrt{2}(2\pi)} \int \rho_\Lambda \left(\frac{p_1 - p_2}{2} \right) \chi^*(p_1 + p_2) a(p_1) a(p_2) \, dp_1 \, dp_2. \quad (47)$$

Then B_Λ is an operator from \mathcal{H} to $\tilde{\mathcal{H}}$, and the key point is that it provides a square root for $H_{I,\Lambda}$.

Lemma 1. For $\Lambda < \infty$

1. B_Λ defines a bounded operator on each subspace \mathcal{H}_N .
2. For $\psi \in \mathcal{D}$ we have in the representation (45)

$$(B_\Lambda \psi)(p) = \frac{1}{\sqrt{2}(2\pi)} \int \rho_\Lambda(q) a\left(\frac{p}{2} + q\right) a\left(\frac{p}{2} - q\right) \psi \, dq. \quad (48)$$

3. On each \mathcal{H}_N

$$-g_\Lambda B_\Lambda^* B_\Lambda = H_{I,\Lambda}. \quad (49)$$

Proof. The expression is naturally defined as a bilinear form. For $\psi \in \mathcal{D}$ and $\Psi \in \tilde{\mathcal{D}}$ we have in the representation (45)

$$(\Psi, B_\Lambda \psi) = \frac{1}{\sqrt{2}(2\pi)} \int \rho_\Lambda \left(\frac{p_1 - p_2}{2} \right) (\Psi(p_1 + p_2), a(p_1) a(p_2) \psi) \, dp_1 \, dp_2. \quad (50)$$

Applying the Schwarz inequality twice we have

$$\begin{aligned}
 |(\Psi, B_\Lambda \psi)| &\leq \left(\int \left| \rho_\Lambda \left(\frac{p_1 - p_2}{2} \right) \right|^2 \|\Psi(p_1 + p_2)\|^2 dp_1 dp_2 \right)^{1/2} \\
 &\quad \times \left(\int \|a(p_1)a(p_2)\psi\|^2 dp_1 dp_2 \right)^{1/2} \\
 &\leq \|\rho_\Lambda\|_2 \|\Psi\| \|N_0\psi\|.
 \end{aligned}
 \tag{51}$$

Now specialize to $\psi \in \mathcal{H}_N$ and we see that B_Λ is a bounded bilinear form and hence a bounded operator. This establishes the first point.

Next change variables in (50) and obtain

$$(\Psi, B_\Lambda \psi) = \frac{1}{\sqrt{2}(2\pi)} \int \rho_\Lambda(q) \left(\Psi(p), a\left(\frac{p}{2} + q\right) a\left(\frac{p}{2} - q\right) \psi \right) dp dq.
 \tag{52}$$

which establishes (48).

For (49) it suffices to establish the identity as a quadratic form on \mathcal{D} . Inserting the representation of (48) into $-g_\Lambda \|B_\Lambda \psi\|^2$ we obtain the representation (40) of $(\psi, H_{I,\Lambda} \psi)$. This completes the proof. \square

For later reference we consider the case $\Lambda = \infty$ with the operator

$$B = \frac{1}{\sqrt{2}(2\pi)} \int \chi^*(p_1 + p_2) a(p_1) a(p_2) dp_1 dp_2.
 \tag{53}$$

Lemma 2. *B defines an (unbounded) operator on $\mathcal{H}_N \cap D(H_0)$ which satisfies for some constant C:*

$$\|B\psi\| \leq C \|(H_0 + N)\psi\|.
 \tag{54}$$

For ψ in this domain

$$\lim_{\Lambda \rightarrow \infty} B_\Lambda \psi = B\psi.
 \tag{55}$$

Proof. All the above representations still hold for $\psi \in \mathcal{D}$, $\Psi \in \tilde{\mathcal{D}}$. But now instead of (51) we have

$$\begin{aligned}
 |(\Psi, B\psi)| &\leq \left(\int (\omega(p_1) + 1)^{-1} (\omega(p_2) + 1)^{-1} \|\Psi(p_1 + p_2)\|^2 dp_1 dp_2 \right)^{1/2} \\
 &\quad \times \left(\int (\omega(p_1) + 1) (\omega(p_2) + 1) \|a(p_1)a(p_2)\psi\|^2 dp_1 dp_2 \right)^{1/2} \\
 &\leq \left(\int \left(\omega\left(\frac{p}{2} + q + 1\right) \right)^{-1} \left(\omega\left(\frac{p}{2} - q + 1\right) \right)^{-1} \|\Psi(p)\|^2 dp dq \right)^{1/2} \| (H_0 + N_0)\psi \| \\
 &\leq C \|\Psi\| \| (H_0 + N_0)\psi \|.
 \end{aligned}
 \tag{56}$$

Here $C = \left(\int (\omega(q) + 1)^{-2} dq \right)^{1/2}$ and in the last step we use the Schwarz inequality in q . This shows that B defines an operator on $\mathcal{D} \cap \mathcal{H}_N$ satisfying inequality (54). Since $\mathcal{D} \cap \mathcal{H}_N$ is a core for H_0 on \mathcal{H}_N we can extend the domain to $D(H_0) \cap \mathcal{H}_N$.

For the second point we estimate $|(\Psi, (B - B_\Lambda)\psi)|$ as above. In the last integral over q we are now restricting to $|q| \geq \Lambda$. Break this into two terms using

$$\{q : |q| \geq \Lambda\} \subset \left\{ q : \left| \frac{p}{2} + q \right| \geq \frac{\Lambda}{2} \right\} \cup \left\{ q : \left| \frac{p}{2} - q \right| \geq \frac{\Lambda}{2} \right\}.
 \tag{57}$$

With $\delta C_\Lambda = \left(\int_{q \geq \Lambda} (\omega(q) + 1)^{-2} dq\right)^{1/2}$ and $\psi \in \mathcal{D} \cap \mathcal{H}_N$ this leads

$$|(\Psi, (B_\Lambda - B)\psi)| \leq \sqrt{2\delta C_\Lambda C} \|\Psi\| \|(H_0 + N)\psi\|. \quad (58)$$

This estimate extends to $\psi \in D(H_0) \cap \mathcal{H}_N$. Then as $\Lambda \rightarrow 0$ we have $\delta C_\Lambda \rightarrow 0$ and $\|(B_\Lambda - B)\psi\| \rightarrow 0$. \square

3.3. Resolvents

We return to $\Lambda < \infty$ and work out some consequences of the identity (49) for resolvents. We define

$$R_0(E) = (H_0 - E)^{-1} \quad R_\Lambda(E) = (H_\Lambda - E)^{-1}. \quad (59)$$

These exist for $\text{Im}E \neq 0$ and $R_0(E)$ exists for $E < 0$. We want to find real E such that $R_\Lambda(E)$ exists as a means to isolate the spectrum of H_Λ .

To this end we also introduce the operators on $\mathcal{H} \oplus \tilde{\mathcal{H}}$

$$\tilde{H}_\Lambda(E) = \begin{pmatrix} H_0 - E & B_\Lambda^* \\ B_\Lambda & g_\Lambda^{-1} \end{pmatrix} \quad \tilde{R}_\Lambda(E) = \tilde{H}_\Lambda(E)^{-1}. \quad (60)$$

Since B_Λ is a bounded operator from \mathcal{H}_N to $\tilde{\mathcal{H}}_{N-2}$ we have that $\tilde{H}_\Lambda(E)$ preserves the subspace $\mathcal{H}_N \oplus \tilde{\mathcal{H}}_{N-2}$. More precisely it is defined on $(D(H_0) \cap \mathcal{H}_N) \oplus \tilde{\mathcal{H}}_{N-2}$ and is self-adjoint there.

Lemma 3. For $E < 0$, $R_\Lambda(E)$ exists in $\mathcal{B}(\mathcal{H}_N)$ iff $\tilde{R}_\Lambda(E)$ exists in $\mathcal{B}(\mathcal{H}_N \oplus \tilde{\mathcal{H}}_{N-2})$ in which case

$$\tilde{R}_\Lambda(E) = \begin{pmatrix} R_\Lambda(E) & -g_\Lambda R_\Lambda(E) B_\Lambda^* \\ -g_\Lambda B_\Lambda R_\Lambda(E) & g_\Lambda + g_\Lambda^2 B_\Lambda R_\Lambda(E) B_\Lambda^* \end{pmatrix}. \quad (61)$$

Proof. We omit the subscript Λ for the proof. First assume that $\tilde{R}(E)$ exists. Then it is self-adjoint and has the form

$$\tilde{R}(E) = \begin{pmatrix} \alpha & \beta^* \\ \beta & \delta \end{pmatrix} \quad (62)$$

for bounded α, β, δ and α, δ self-adjoint. The statement that is the inverse says that α, β^* map into the domain of H_0 and that

$$\begin{aligned} (H_0 - E)\alpha + B^*\beta &= I & (H_0 - E)\beta^* + B^*\delta &= 0 \\ B\alpha + g^{-1}\beta &= O & B\beta^* + g^{-1}\delta &= I. \end{aligned} \quad (63)$$

We ignore the second equation. The third equation says

$$\beta = -gB\alpha \quad \beta^* = -g\alpha B^*. \quad (64)$$

Inserting the expression for β into the first equation and using $-gB^*B = H_I$ we get $(H - E)\alpha = I$. Hence $R(E)$ exists and equals α . Inserting the expression for β^* into the last equation gives $\delta = g + g^2gBR(E)B^*$

On the other hand, if $R(E)$ exists one can check directly that (61) provides a bounded inverse. This completes the proof. \square

Now we give another version.

Definition 2. For $E < 0$ define a bounded operator on each $\tilde{\mathcal{H}}_N$ by

$$\Phi_\Lambda(E) = g_\Lambda^{-1} - B_\Lambda R_0(E) B_\Lambda^*. \quad (65)$$

Lemma 4. For $E < 0$, $\tilde{R}_\Lambda(E)$ exists in $\mathcal{B}(\mathcal{H}_N \oplus \tilde{\mathcal{H}}_{N-2})$ iff $\Phi_\Lambda(E)^{-1}$ exists in $\mathcal{B}(\tilde{\mathcal{H}}_{N-2})$ in which case

$$\tilde{R}_\Lambda(E) = \begin{pmatrix} R_0(E) + R_0(E)B_\Lambda^*\Phi_\Lambda(E)^{-1}B_\Lambda R_0(E) & -R_0(E)B_\Lambda^*\Phi_\Lambda(E)^{-1} \\ -\Phi_\Lambda(E)^{-1}B_\Lambda R_0(E) & \Phi_\Lambda(E)^{-1} \end{pmatrix}. \tag{66}$$

Proof. Again suppose that $\tilde{R}(E)$ exists so we must solve equations (63) again. This time we ignore the third equation. Then the second equation says that

$$\beta^* = -R_0(E)B^*\delta \quad \beta = -\delta BR_0(E). \tag{67}$$

Substituting β^* into the fourth equation gives $(-BR_0(E)B^* + g^{-1})\delta = I$ or $\Phi(E)\delta = I$. Hence $\Phi(E)^{-1}$ exists and equals δ . Substituting β into the first equation gives $(H_0 - E)\alpha - B^*\Phi(E)^{-1}BR_0(E) = I$ whence $\alpha = R_0(E) + R_0(E)B^*\Phi(E)^{-1}BR_0(E)$.

On the other hand, if $\Phi(E)^{-1}$ exists one can check directly that (66) provides a bounded inverse. This completes the proof. \square

Comparing these results we have

Lemma 5. For $E < 0$, $R_\Lambda(E)$ exists in $\mathcal{B}(\mathcal{H}_N)$ iff $\Phi_\Lambda(E)^{-1}$ exists in $\mathcal{B}(\tilde{\mathcal{H}}_{N-2})$ in which case

$$\begin{aligned} R_\Lambda(E) &= R_0(E) + R_0(E)B_\Lambda^*\Phi_\Lambda(E)^{-1}B_\Lambda R_0(E) \\ \Phi_\Lambda(E)^{-1} &= g_\Lambda + g_\Lambda^2 B_\Lambda R_\Lambda(E)B_\Lambda^*. \end{aligned} \tag{68}$$

3.4. Renormalization

In view of the last result we can study the resolvent $R_\Lambda(E)$ on \mathcal{H}_N by studying the operator $\Phi_\Lambda(E)$ on $\tilde{\mathcal{H}}_{N-2}$. The advantage of this operator is that it can be more easily renormalized.

First we Wick-order moving creation operators to the left and annihilation operators to the right using $[a(p), a^*(p')] = \delta(p - p')$ and

$$(H_0 - E)^{-1}a^*(p) = a^*(p)(H_0 + \omega(p) - E)^{-1}. \tag{69}$$

The resulting identity is formal but a rigorous version can be had by regularizing $a(p) \rightarrow a(\delta_\kappa(\cdot - p))$ with approximate delta functions δ_κ . We find

$$\Phi_\Lambda(E) = \Phi_{0,\Lambda}(E) + \Phi_{I,\Lambda}(E) \tag{70}$$

where

$$\begin{aligned} \Phi_{0,\Lambda}(E) &= g_\Lambda^{-1} - \frac{1}{2(2\pi)^2} \int dp_1 dp_2 \chi^*(p_1 + p_2)\chi(p_1 + p_2) \\ &\quad \times \rho_\Lambda \left(\frac{p_1 - p_2}{2} \right)^2 \frac{2}{H_0 + \omega_1 + \omega_2 - E} \\ \Phi_{I,\Lambda}(E) &= -\frac{1}{2(2\pi)^2} \int dp_1 dp_2 dp'_1 dp'_2 \chi^*(p_1 + p_2)\chi(p'_1 + p'_2)\rho_\Lambda \left(\frac{p_1 - p_2}{2} \right) \\ &\quad \times \rho_\Lambda \left(\frac{p'_1 - p'_2}{2} \right) \left(a^*(p'_1)a^*(p'_2) \frac{1}{H_0 + \omega_1 + \omega_2 + \omega'_1 + \omega'_2 - E} a(p_1)a(p_2) \right. \\ &\quad \left. + \delta(p_1 - p'_1)a^*(p'_2) \frac{4}{H_0 + \omega_1 + \omega_2 + \omega'_2 - E} a(p_2) \right). \end{aligned} \tag{71}$$

Here $\omega_1 = \omega(p_1) = p_1^2/2$, etc. These are bilinear forms on $\tilde{\mathcal{D}} \times \tilde{\mathcal{D}}$. By the methods of section 3.2 they determine bounded operators on each $\tilde{\mathcal{H}}_{N-2}$ for $\Lambda < \infty$. But now we want to work uniformly in Λ and also include $\Lambda = \infty$.

To cancel the divergence in $\Phi_{0,\Lambda}(E)$ we change variables and write

$$\Phi_{0,\Lambda}(E) = (2\pi)^{-2} \left(\int_{|q| \leq \Lambda} (q^2 + \mu^2)^{-1} - \int_{|q| \leq \Lambda} dp dq \chi^*(p) \chi(p) \frac{1}{H_0 + p^2/4 + q^2 - E} \right). \quad (72)$$

In the representation $\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}^2, \tilde{\mathcal{H}})$ this is⁵

$$\begin{aligned} (\Phi_{0,\Lambda}(E)\Psi)(p) &= (2\pi)^{-2} \left(\int_{|q| \leq \Lambda} (q^2 + \mu^2)^{-1} - \int_{|q| \leq \Lambda} dq \frac{1}{H_0 + p^2/4 + q^2 - E} \right) \Psi(p) \\ &= (2\pi)^{-2} \xi_\Lambda(\mu^2, H_0 + p^2/4 - E) \Psi(p). \end{aligned} \quad (73)$$

As noted in (14), ξ_Λ has no divergence and we can define for $\Lambda = \infty$:

$$\begin{aligned} (\Phi_0(E)\Psi)(p) &= (2\pi)^{-2} \xi(\mu^2, H_0 + p^2/4 - E) \Psi(p) \\ &= (4\pi)^{-1} \log \left(\frac{H_0 + p^2/4 - E}{\mu^2} \right) \Psi(p). \end{aligned} \quad (74)$$

Lemma 6. For $E < -\mu^2$, $\Phi_0(E)$ is essentially self-adjoint on $\tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$ and for Ψ in this domain we have

$$\lim_{\Lambda \rightarrow \infty} \Phi_{0,\Lambda}(E)\Psi = \Phi_0(E)\Psi. \quad (75)$$

Proof. For the essential self-adjointness it suffices to show that the domain contains a dense set of analytic vectors (Nelson's theorem, [8]). For analytic vectors we can take wavefunctions with compact support.

The convergence is straightforward. One can use the inequality

$$\begin{aligned} &\left\| \left(\log \left(H_0 + p^2/4 - E + \frac{1}{\Lambda^2} \right) - \log(H_0 + p^2/4 - E) \right) \Psi(p) \right\| \\ &\leq \Lambda^{-2} \|(H_0 + p^2/4 - E)^{-1} \Psi(p)\| \leq \mathcal{O}(\Lambda^{-2}) \|\Psi(p)\| \end{aligned} \quad (76)$$

which follows using the spectral theorem. \square

Next, we work on $\Phi_{I,\Lambda}(E)$. For $\Lambda = \infty$ it is defined without the ρ_Λ and denoted by $\Phi_I(E)$.

Lemma 7. For $E < -1$ and $\Lambda \leq \infty$ and $\Psi \in \tilde{\mathcal{D}}$:

$$|(\Psi, \Phi_{I,\Lambda}(E)\Psi)| \leq 2(\Psi, N_0^2\Psi). \quad (77)$$

Thus $\Phi_{I,\Lambda}(E)$, $\Phi_I(E)$ define bounded operators on $\tilde{\mathcal{H}}_N$ and for $\Psi \in \tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$:

$$\lim_{\Lambda \rightarrow \infty} \Phi_{I,\Lambda}(E)\Psi = \Phi_I(E)\Psi. \quad (78)$$

Proof. We take $\Phi_{I,\Lambda}(E) = \Phi_{I,\Lambda}^{(2)}(E) + \Phi_{I,\Lambda}^{(4)}(E)$ where the superscript indicates the number of creation and annihilation operators. For the first we have

$$\begin{aligned} |(\Psi, \Phi_{I,\Lambda}^{(2)}(E)\Psi)| &\leq \frac{1}{2\pi^2} \int dp_1 dp_2 dp'_2 \\ &\times \left| \left(a(p'_2)\Psi(p_1 + p_2), \frac{1}{H_0 + \omega_1 + \omega_2 + \omega'_2 - E} a(p_2)\Psi(p_1 + p'_2) \right) \right| \end{aligned}$$

⁵ In general, if $T = \int \chi^*(p)T(p)\chi(p) dp$ defines an operator on $\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}^2) \otimes \mathcal{H}$, then in the representation $\tilde{\mathcal{H}} = \mathcal{L}^2(\mathbb{R}^2, \mathcal{H})$ we have $(T\Psi)(p) = T(p)\Psi(p)$.

$$\begin{aligned}
 &\leq \frac{1}{2\pi^2} \int dp_1 dp_2 dp'_2 \|a(p'_2)\Psi(p_1 + p_2)\| \frac{1}{\omega_2 + \omega'_2 + 1} \|a(p_2)\Psi(p_1 + p'_2)\| \\
 &\leq \frac{1}{2\pi^2} \int dp_2 dp'_2 \|a(p'_2)\Psi\| \frac{1}{\omega_2 + \omega'_2 + 1} \|a(p_2)\Psi\| \\
 &\leq \|N_0^{1/2}\Psi\|^2.
 \end{aligned} \tag{79}$$

Here in the last step we use the fact, noted in [4], that for $h, h' \in \mathcal{L}^2(\mathbb{R}^2)$ and any $c > 0$:

$$\left| \int h(p) \frac{1}{p^2 + q^2 + c} h'(q) dp dq \right| \leq \pi^2 \|h\|_2 \|h'\|_2. \tag{80}$$

For the convergence we proceed differently. We use the estimate for $\epsilon > 0$

$$\left| \rho_\Lambda \left(\frac{p_1 - p_2}{2} \right) \rho_\Lambda \left(\frac{p'_1 - p'_2}{2} \right) - 1 \right| \leq \mathcal{O}(\Lambda^{-\epsilon})(\omega_1 + \omega_2 + \omega'_2 + 1)^\epsilon. \tag{81}$$

Then for $\Psi_1, \Psi_2 \in \tilde{\mathcal{D}}$ we have

$$\begin{aligned}
 |(\Psi_1, (\Phi_{I,\Lambda}^{(2)}(E) - \Phi_I^{(2)}(E))\Psi_2)| &\leq \mathcal{O}(\Lambda^{-\epsilon}) \int dp_1 dp_2 dp'_2 \|a(p'_2)\Psi_1(p_1 + p_2)\| \\
 &\quad \times \frac{1}{(\omega_1 + \omega_2 + \omega'_2 + 1)^{1-\epsilon}} \|a(p_2)\Psi_2(p_1 + p'_2)\| \\
 &\leq \mathcal{O}(\Lambda^{-\epsilon}) \int dp_1 dp_2 dp'_2 \frac{1}{(\omega_2 + 1)^{\frac{3}{4}-\epsilon/2}} \|a(p'_2)\Psi_1(p_1 + p_2)\| \\
 &\quad \times \frac{(\omega_2 + 1)^{1/2}}{(\omega'_2 + 1)^{\frac{3}{4}-\epsilon/2}} \|a(p_2)\Psi_2(p_1 + p'_2)\| \\
 &\leq \mathcal{O}(\Lambda^{-\epsilon}) \|N_0^{\frac{1}{2}}\Psi_1\| \|(H_0 + N_0)^{\frac{1}{2}}\Psi_2\|
 \end{aligned} \tag{82}$$

where the last step follows by the Schwarz inequality. Specializing to $\tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$ the estimate is uniform in $\|\Psi_1\| = 1$ and yields the convergence $\|(\Phi_{I,\Lambda}^{(2)}(E) - \Phi_I^{(2)}(E))\Psi_2\| \rightarrow 0$ (in fact strong convergence holds since we have a uniform bound on the norms).

For the second term we define

$$f(p', q', p) = \left\| a \left(\frac{p'}{2} + q' \right) a \left(\frac{p'}{2} - q' \right) \Psi(p) \right\| \tag{83}$$

and find

$$\begin{aligned}
 (\Psi, \Phi_{I,\Lambda}^{(4)}(E)\Psi) &\leq \frac{1}{8\pi^2} \int dp_1 dp_2 dp'_1 dp'_2 \|a(p'_1)a(p'_2)\Psi(p_1 + p_2)\| \\
 &\quad \times \frac{1}{\omega_1 + \omega_2 + \omega'_1 + \omega'_2 + 1} \|a(p_1)a(p_2)\Psi(p'_1 + p'_2)\| \\
 &\leq \frac{1}{8\pi^2} \int dp dq dp' dq' f(p', q', p) \frac{1}{q^2 + (q')^2 + 1} f(p, q, p') \\
 &\leq \frac{1}{8} \int dp dp' \|f(p', \cdot, p)\|_2 \|f(p, \cdot, p')\|_2 \\
 &\leq \frac{1}{8} \|f\|_2^2 \leq \frac{1}{8} \|N_0\Psi\|^2.
 \end{aligned} \tag{84}$$

Again we have used (80). This completes the bound, and the convergence follows by an estimate similar to (82). \square

To combine these we have

Lemma 8.

1. For $E < -1$, $\Phi(E)$ is essentially self-adjoint on $\tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$ and for Ψ in this domain

$$\lim_{\Lambda \rightarrow \infty} \Phi_\Lambda(E)\Psi = \Phi(E)\Psi. \quad (85)$$

2. Let $E < -e_N$ where

$$e_N = \max(1, \mu^2 e^{16\pi N^2}). \quad (86)$$

Then for Λ sufficiently large or $\Lambda = \infty$ we have that $\Phi_\Lambda(E)$ is strictly positive and for $\Psi \in \tilde{\mathcal{H}}_N$

$$\lim_{\Lambda \rightarrow \infty} \Phi_\Lambda(E)^{-1}\Psi = \Phi(E)^{-1}\Psi. \quad (87)$$

Proof. $\Phi(E) = \Phi_0(E) + \Phi_I(E)$ is the sum of an essentially self-adjoint operator and a bounded operator. The essential self-adjointness again follows by Kato's theorem. The convergence follows from our results (75), (78).

For the second part under our assumptions $\xi(\mu^2, -E) = \pi \log(-E/\mu^2) \geq 16\pi^2 N^2$. Then since $\xi_\Lambda(\mu^2, -E)$ converges to $\xi(\mu^2, -E)$ we have for Λ sufficiently large (depending on E, μ) $\xi_\Lambda(\mu^2, -E) \geq 12\pi^2 N^2$. Since $\xi_\Lambda(a, b)$ is increasing in b we have for $\Psi \in \tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$

$$(\Psi, \Phi_{0,\Lambda}(E)\Psi) \geq (2\pi)^{-2} \xi_\Lambda(\mu^2, -E) \|\Psi\|^2 \geq 3N^2 \|\Psi\|^2. \quad (88)$$

Combining this with the bound $|(\Psi, \Phi_{I,\Lambda}(E)\Psi)| \leq 2N^2 \|\Psi\|^2$ we have for Λ sufficiently large or $\Lambda = \infty$:

$$(\Psi, \Phi_\Lambda(E)\Psi) \geq N^2 \|\Psi\|^2. \quad (89)$$

This gives the positivity and shows that $\Phi_\Lambda(E)$ has a bounded inverse. Convergence on the core $\tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$ for $\Phi(E)$ and the uniform bound $\|\Phi_\Lambda(E)^{-1}\| \leq N^{-2}$ imply the strong convergence for $\Phi_\Lambda(E)^{-1}$ (see for example [6], p 429). \square

3.5. Resolvent convergence

Now we can prove the main result (cf Dell'Antonio *et al* [4])

Theorem 3.

1. For real $E < -e_N$ and $\psi \in \mathcal{H}_N$ the limit $R(E)\psi = \lim_{\Lambda \rightarrow \infty} R_\Lambda(E)\psi$ exists and is equal to

$$R(E) = R_0(E) + R_0(E)B^*\Phi(E)^{-1}BR_0(E). \quad (90)$$

2. $R(E)$ is invertible.
 3. For E complex and not in $[-e_N, \infty)$ the limit $R(E)\psi = \lim_{\Lambda \rightarrow \infty} R_\Lambda(E)\psi$ exists. There is a self-adjoint operator $H(\mu)$ with $H(\mu) \geq -e_N$ so $R(E) = (H(\mu) - E)^{-1}$.

Proof.

1. By lemma 8 if $E < -e_N$ and Λ is sufficiently large then $\Phi_\Lambda(E)^{-1}$ exists as a bounded operator on \mathcal{H}_{N-2} . By lemma 5 it follows that all such real E are in the resolvent set of H_Λ on \mathcal{H}_N and

$$R_\Lambda(E) = R_0(E) + R_0(E)B_\Lambda^* \Phi_\Lambda(E)^{-1} B_\Lambda R_0(E). \tag{91}$$

We claim that $B_\Lambda R_0(E)$ converges in norm to $BR_0(E)$. By the resolvent identity it suffices to prove this for any $E < 0$ and we take $E = -N$ and show $B_\Lambda(H_0 + N)^{-1}$ converges in norm to $B(H_0 + N)^{-1}$. This follows by (58). Taking adjoints we have that $R_0(E)B_\Lambda^*$ converges in norm to $R_0(E)B^*$. We also know by lemma 8 that $\Phi_\Lambda(E)^{-1}$ converges strongly to $\Phi(E)^{-1}$. Combining these results we have that $R_\Lambda(E)$ converges strongly to $R(E)$ given by (90).

2. As in the proof of theorem 1 it suffices to find a dense domain of vectors $\psi \in \mathcal{H}_N$ so that $H_\Lambda \psi$ converges. In fact we show $H_{I,\Lambda} \psi \rightarrow 0$ which suffices. We have $H_{I,\Lambda} \psi = -g_\Lambda B_\Lambda^* B_\Lambda \psi$. By (51) $\|B_\Lambda^*\| \leq \|\rho_\Lambda\|_2 N \leq \mathcal{O}(\Lambda)$. Since also $g_\Lambda \rightarrow 0$ suffices to find a dense domain so that $\|B_\Lambda \psi\| = \mathcal{O}(\Lambda^{-1})$.

Now \mathcal{H}_N can be thought of as symmetric functions in $\mathcal{L}^2(\mathbb{R}^{2N})$. We take the subspace of functions in $\mathcal{S}(\mathbb{R}^{2N})$ which have a Fourier transform in $\mathcal{C}_0^\infty(\mathbb{R}^{2N})$ with support disjoint from the hypersurfaces where points coincide. If ψ is in this space then $a(p_1)a(p_2)\psi \sim \psi(p_1, p_2, \dots)$ is a vector-valued function which has a Fourier transform in $\mathcal{C}_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ with support disjoint from the diagonal. Then

$$u(p, q) \equiv \frac{1}{\sqrt{2}(2\pi)} a\left(\frac{p}{2} + q\right) a\left(\frac{p}{2} - q\right) \psi \tag{92}$$

has a Fourier transform $\hat{u}(X, x)$ which is an element of $\mathcal{C}_0^\infty(\mathbb{R}^2 \times (\mathbb{R}^2 - \{0\}))$. Hence $\hat{v} = |x|^{-2} \hat{u}$ is in the same space and if $v(p, q)$ is the inverse Fourier transform then $u = -\Delta_q v$.

Now we have for any n

$$\begin{aligned} (B_\Lambda \psi)(p) &= \int \rho_\Lambda(q) u(p, q) \, dq \\ &= \int \rho_\Lambda^*(q) u(p, q) \, dq + \mathcal{O}(\Lambda^{-n}) \\ &= \int (-\Delta_q \rho_\Lambda^*(q)) v(p, q) \, dq + \mathcal{O}(\Lambda^{-n}). \end{aligned} \tag{93}$$

Here we first replace the sharp cutoff ρ_Λ by a smooth cutoff ρ_Λ^* and then integrate by parts. Since $|\Delta_q \rho_\Lambda^*(q)| = \mathcal{O}(\Lambda^{-2})$ and since $v(p, q)$ is rapidly decreasing in both variables we have $\|B_\Lambda \psi\| = \mathcal{O}(\Lambda^{-2})$ which suffices.

3. This follows by the Trotter–Kato theorem. □

Remarks.

1. For large N our lower bound is $H \geq -\mu^2 e^{16\pi N^2}$. The coefficient 16π can be improved but in any case the N^2 behaviour is probably not optimal. Indeed, mean field calculations [3] suggest that the actual lower bound may be $e^{\mathcal{O}(N)}$. The ground state is presumably a dense clump of particles: a ‘bosonic star’.
2. For further studies of the spectrum one can consider the operator $\Phi(E)^{-1}$. We note that for $E < 0$ if one scales all momenta by $\sqrt{-E}$ the operator $\Phi(E)$ becomes

$$\frac{1}{4\pi} \log\left(\frac{-E}{\mu^2}\right) + W \tag{94}$$

where

$$\begin{aligned}
W = & (4\pi)^{-1} \int dp \chi^*(p) \log(H_0 + p^2/4 + 1) \chi(p) \\
& - \frac{1}{2(2\pi)^2} \int dp_1 dp_2 dp'_1 dp'_2 \chi^*(p_1 + p_2) \chi(p'_1 + p'_2) \\
& \times \left(a^*(p'_1) a^*(p'_2) \frac{1}{H_0 + \omega_1 + \omega_2 + \omega'_1 + \omega'_2 + 1} a(p_1) a(p_2) \right. \\
& \left. + \delta(p_1 - p'_1) a^*(p'_2) \frac{4}{H_0 + \omega_1 + \omega_2 + \omega'_2 + 1} a(p_2) \right). \tag{95}
\end{aligned}$$

The issue is then to study properties of W .

3.6. Extensions

We now allow an extra inter-particle potential v again assumed bounded. This means we add a potential

$$V = \frac{1}{2} \int a^*(x) a^*(y) v(x - y) a(x) a(y) dx dy. \tag{96}$$

We have

$$|(\psi, V\chi)| \leq \frac{1}{2} \|v\|_\infty \|N_0^{1/2}(N_0 - 1)^{1/2}\psi\| \|N_0^{1/2}(N_0 - 1)^{1/2}\chi\| \tag{97}$$

and thus V defines an operator on \mathcal{H}_N satisfying $\|V\| \leq N^2 \|v\|_\infty / 2$. This is in configuration space and we actually consider the momentum space version $V' = \Gamma(\mathcal{F})V\Gamma(\mathcal{F}^{-1})$ where $\Gamma(\mathcal{F})$ is the induced Fourier transform on Fock space. This also satisfies $\|V'\| \leq N^2 \|v\|_\infty / 2$ which is the only fact we use.

With a cutoff the full Hamiltonian is then

$$H_\Lambda^\# = H_0 + V' + H_{I,\Lambda}. \tag{98}$$

Then $H_\Lambda^\#$ is self-adjoint on $D(H_0) \cap \mathcal{H}_N$. The same is true for

$$H_1 = H_0 + V' \tag{99}$$

and we have $H_1 \geq -N^2 \|v\|_\infty / 2$

Proceeding as before we introduce resolvents

$$R_1(E) = (H_1 - E)^{-1} \quad R_\Lambda^\#(E) = (H_\Lambda^\# - E)^{-1} \tag{100}$$

and for $E < -N^2 \|v\|_\infty / 2$

$$\Phi_\Lambda^\#(E) = g_\Lambda^{-1} - B_\Lambda R_1(E) B_\Lambda^*. \tag{101}$$

For such E we find as in lemma 5 that E is in the resolvent set of $H_\Lambda^\#$ on \mathcal{H}_N if and only if $\Phi_\Lambda^\#(E)$ has a bounded inverse on $\tilde{\mathcal{H}}_{N-2}$ in which case

$$R_\Lambda^\#(E) = R_1(E) + R_1(E) B_\Lambda^* \Phi_\Lambda^\#(E)^{-1} B_\Lambda R_1(E). \tag{102}$$

Theorem 4.

1. Let $E < -e_N^\#$ where

$$e_N^\# = \max(N, N^2 \|v\|_\infty, -\mu^2 e^{16\pi N^2 (C^2 \|v\|_\infty + 1)}) \tag{103}$$

and where C is the constant in lemma 2. For $\psi \in \mathcal{H}_N$ the limit $R^\#(E)\psi = \lim_{\Lambda \rightarrow \infty} R_\Lambda^\#(E)\psi$ exists and is equal to

$$R^\#(E) = R_1(E) + R_1(E) B^* \Phi^\#(E)^{-1} B R_1(E). \tag{104}$$

2. $R^\#(E)$ is invertible.
3. For E complex and not in $[-e_N^\#, \infty)$ the limit $R^\#(E)\psi = \lim_{\Lambda \rightarrow \infty} R_\Lambda^\#(E)\psi$ exists. There is a self-adjoint operator $H^\#(\mu)$ with $H^\#(\mu) \geq -e_N^\#$ so that $R^\#(E) = (H^\#(\mu) - E)^{-1}$.

Proof. We follow the proof of theorem 3. We have

$$R_1(E) = R_0(E) - R_1(E)V'R_0(E) \tag{105}$$

and hence

$$\Phi_\Lambda^\#(E) = \Phi_\Lambda(E) + B_\Lambda R_0(E)V'R_0(E)B_\Lambda^* - B_\Lambda R_0(E)V'R_1(E)V'R_0(E)B_\Lambda^*. \tag{106}$$

For $\Lambda = \infty$ define $\Phi^\#(E)$ by replacing $\Phi_\Lambda(E)$ by $\Phi(E)$ and B_Λ by B . Since $E < -\mu^2 e^{16\pi N^2(C^2\|v\|_\infty+1)}$ we have for Λ sufficiently large or infinite instead of (88)

$$\Phi_{0,\Lambda}(E) \geq 3N^2(C^2\|v\|_\infty + 1) \tag{107}$$

and it follows by the bound on $\Phi_{l,\Lambda}(E)$ that

$$\Phi_\Lambda(E) \geq N^2(C^2\|v\|_\infty + 1). \tag{108}$$

For the other terms in (106) we note that $E < -N$ implies $\|B_\Lambda R_0(E)\| \leq C$ by lemma 2. Also $E < -N^2\|v\|_\infty$ and the lower bound on H_1 imply that $\|R_1(E)\| \leq (N^2\|v\|_\infty/2)^{-1}$. Using also $\|V'\| \leq N^2\|v\|_\infty/2$

$$\begin{aligned} \|B_\Lambda R_0(E)V'R_0(E)B_\Lambda^*\| &\leq C^2N^2\|v\|_\infty/2 \\ \|B_\Lambda R_0(E)V'R_1(E)V'R_0(E)B_\Lambda^*\| &\leq C^2N^2\|v\|_\infty/2. \end{aligned} \tag{109}$$

Combining these we find for $\Lambda \leq \infty$

$$\Phi_\Lambda^\#(E) \geq N^2 \tag{110}$$

so that $\Phi_\Lambda(E)^{-1}$ exists. Then for $\Lambda < \infty$ all $E < -e_N^\#$ are in the resolvent set for $R_\Lambda^\#(E)$ and (102) holds.

As before $\Phi^\#(E)$ is essentially self-adjoint on $\tilde{\mathcal{D}} \cap \tilde{\mathcal{H}}_N$. On this domain $\Phi^\#(E)\psi = \lim_{\Lambda \rightarrow \infty} \Phi_\Lambda^\#(E)\psi$. This follows from the convergence for $\Phi_\Lambda(E)$ and the norm convergence of $B_\Lambda R_0(E)$. Using the uniform bounds on the inverses $\Phi_\Lambda^\#(E)^{-1}$ converges strongly to $\Phi^\#(E)^{-1}$.

Finally $R_\Lambda^\#(E)$ given by (102) converges strongly to $R_\Lambda^\#(E)$ given by (104). Here we use the norm convergence of $B_\Lambda R_1(E)$ to $BR_1(E)$ which can be demonstrated using the adjoint of (105). This completes the proof of the first part and the second and third parts follow as in theorem 3. □

4. Conclusion and discussion

We have established that the N -body Schrödinger operator with delta-function potentials defines a self-adjoint operator, and have obtained an explicit formula for the resolvent. This rederives known results by a new method. We have also established the self-adjointness if the potentials have the form (delta-function) + (bounded), and have obtained an explicit formula for the resolvent in this case. This is a new result.

This work is a contribution to the theory of non-relativistic Schrödinger operators. However, some of the techniques are borrowed from quantum field theory. The work also contributes to the legitimation of renormalization techniques used by physicists in quantum field theory. Finally it is our hope that this work will contribute to further progress on renormalization problems in quantum field theory. We are particularly thinking of Hamiltonian formulations of Yang–Mills theory where similar problems occur.

Appendix. Trotter–Kato theorem

In the text we use the following version of the Trotter–Kato theorem.

Theorem 5. *Let Σ be a proper closed subset of \mathbb{R} and let H_n be a sequence of self-adjoint operators with resolvents $R_n(E) = (H_n - E)^{-1}$ defined for all complex $E \notin \Sigma$. Suppose $R_n(E)$ converges strongly for some $E \notin \Sigma$ and that the limit is invertible. Then there exists a self-adjoint operator H with resolvents $R(E) = (H - E)^{-1}$ such that $R_n(E)$ converges strongly to $R(E)$ for all complex $E \notin \Sigma$.*

A slightly different result is proved in [9]. There $\Sigma = \mathbb{R}$ is allowed, but one needs convergence at two points with $\pm \text{Im } E > 0$. This proof can be easily adapted to prove the quoted result.

References

- [1] Albeverio S, Gesztesy F, Hoegh-Krohn R and Holden H 1988 *Solvable Models in Quantum Mechanics* (Berlin: Springer)
- [2] Henderson R J and Rajeev S G 1998 Renormalized contact potential in two-dimensions *J. Math. Phys.* **39** 749
- [3] Rajeev S G 1999 Bound states in models of asymptotic freedom *Preprint* hep-th/9902025
- [4] Dell’Antonio G F, Figari R and Teta A 1994 Hamiltonians for N particles interacting through point interactions *Ann. Inst. H Poincaré* **60** 253–90
- [5] Flammand G 1967 *Mathematical Theory of Non-Relativistic Two- and Three-Particle Systems with Point Interactions (Cargese Lectures in Theoretical Physics)* ed F Lurcat (London: Gordon and Breach)
- [6] Kato T 1966 *Perturbation Theory for Linear Operators* (Berlin: Springer)
- [7] Huang K 1982 *Quarks, Leptons, and Gauge Fields* (Singapore: World Scientific)
- [8] Reed M and Simon B 1975 *Methods of Modern Mathematical Physics* vol 2 (New York: Academic)
- [9] Reed M and Simon B 1972 *Methods of Modern Mathematical Physics* vol 1 (New York: Academic)